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1995 J. Phys. A: Math. Gen. 28 2729

(http://iopscience.iop.org/0305-4470/28/10/006)

J. Phys. A: Math. Gen. 28 (1995) 2729–2735. Printed in the UK

Bond percolation on branching Koch curves

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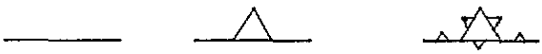
Received 18 October 1994

Abstract. With use of the exact recursion equations of the generating functions, we determine the asymptotic behaviour of bond percolation on branching Koch curves and obtain the critical exponents α, β, γ and ν. The scaling law $2-\alpha=\gamma+2\beta=d_f\nu$ is obtained; this is similar to the scaling law for a regular lattice, except that the fractal dimension d_f is replaced by the Euclidean dimension d.

1. Introduction

Percolation is a useful model for many physical problems, such as polymer and diluted magnet systems; the study of fundamental phase transition models is always an active field in statistical physics [1]. In recent years, percolation on fractals has been studied [2–6]. The results of research have shown that the percolating state takes place with a non-trivial threshold ($p_c<1$) for infinitely ramified fractals and with $p_c=1$ for finitely ramified fractals. Some critical exponents of percolation on infinitely ramified fractals have also been obtained using renormalization-group methods [3]. However, there has been no work concerning the scaling law for fractals.

In this paper we study bond percolation on the branching Koch curve shown in figure 1. The branching Koch curve, together with other deterministic fractals including the Sierpinski gasket and Sierpinski carpet, are often used as a theoretical 'laboratory' on which many physical problems can be solved [7–12]. On the branching Koch curve the bond percolation threshold has been proved to be a trivial value 1 [2, 3]. However, we are still interested in finding the critical exponents and discussing the scaling law, as has been achieved in the case of the one-dimensional thermal phase transition where only a trivial critical point $T_c=0$ exists [13].



$r=0$ $\quad$ $r=1$ $\quad$ $r=2$

Figure 1. Growth of the branching Koch curve. The first three stages are shown.

We have calculated the critical exponents α, β and γ with use of the generating function method proposed by Dhar [14] in studying self-avoiding random walks. The critical exponent ν is obtained by applying a widely used renormalization-group transformation scheme to study the problems of percolation, lattice animals and self-avoiding random walks [15]. We obtained the scaling law, $2-\alpha=\gamma+2\beta=d_f\nu$, which is the first such

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scaling law for fractals. This scaling law is similar to the one for a regular lattice except that the fractal dimension d_f is replaced by Euclidean dimension d. We have also calculated the critical exponents α, β, γ and ν on other branching Koch curves (shown in figure 2) and found that the scaling law $2-\alpha=\gamma+2\beta=d_f\nu$ is always valid.

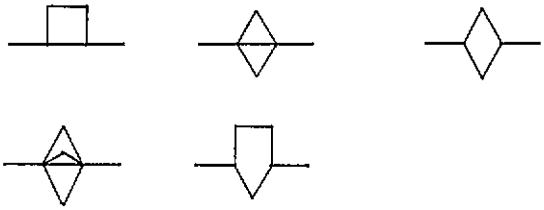

Figure 2. Some branching Koch curves.

2. Method and result

The behaviour of a system close to its percolation threshold is described by critical exponents $\alpha, \beta, \gamma, \nu \ldots$ which are defined as [1]

$$\left[\sum_s n_s(p)\right]_{\rm sing} \propto |p-p_c|^{2-\alpha} \tag{1}$$

$$\left[\sum_s sn_s(p)\right]_{\rm sing} \propto |p-p_c|^{\beta} \tag{2}$$

$$\left[\sum_s s^2n_s(p)\right]_{\rm sing} \propto |p-p_c|^{-\gamma} \tag{3}$$

$$\xi(p) \propto |p-p_c|^{-\nu} \tag{4}$$

where n_s is the average number (per lattice site) of s-bond clusters and depends on concentration p, ξ is the correlation length and corresponds to the average radius of a typical percolation cluster, and the subscript 'sing' denotes the leading singular terms.

The fractal we consider is a branching Koch curve shown in figure 1, with fractal dimension [9]

$$d_f = \frac{\ln 5}{\ln 3}. \tag{5}$$

The key point for finding critical exponents is to form the asymptotic expressions for $\sum_s n_s$, $\sum_s sn_s$ and $\sum_s s^2n_s$. As we see, the structure of the branching Koch curve at stage r is constructed by stage $r-1$ via an iteration procedure; therefore, we can calculate $\sum_s n_s$, $\sum_s sn_s$ and $\sum_s s^2n_s$ stage by stage in terms of recursion relations.

We call a cluster L order r if r is the minimum order at which L can be completely described in the rth-stage branching Koch curve, and denote the number of all rth-order clusters in the rth-stage branching Koch curve by S_r; then we have

$$\sum_s n_s = \frac{1}{N}\sum_r t_r S_r \tag{6}$$

where t_r is the number of the rth-stage branching Koch curve that the whole lattice is divided into. In appendix A, we derive $N=\frac{3}{4}t_r 5^r$, so then equation (6) becomes

$$\sum_s n_s = \frac{4}{3}\sum_r 5^{-r} S_r. \tag{7}$$

We now define three independent restricted generating functions A_r, B_r and D_r which are necessary to construct S_r, as shown in figure 3. The generating function A_r corresponds to those configurations where a part of the cluster penetrates through one corner vertex of the rth-stage branching Koch curve and terminates at its internal points. B_r corresponds to those configurations where a part of the cluster joins the two corner vertices of the rth-stage branching Koch curve. D_r corresponds to those configurations where two parts of the cluster penetrate separately through the two corner vertices but do not join each other. In figure 4 we have drawn all possible rth-order clusters; S_r can now be written as

$$S_r = (1 + B_{r-1}^3)A_{r-1}^2 + (2B_{r-1} + 2)A_{r-1}^3 + 3B_{r-1}^2 A_{r-1}^2 D_{r-1} + B_{r-1}A_{r-1}^4. \tag{8}$$

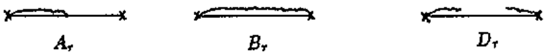

Figure 3. Three restricted generating functions are shown. 'x' denotes the corner vertex of the rth-stage Koch curve.

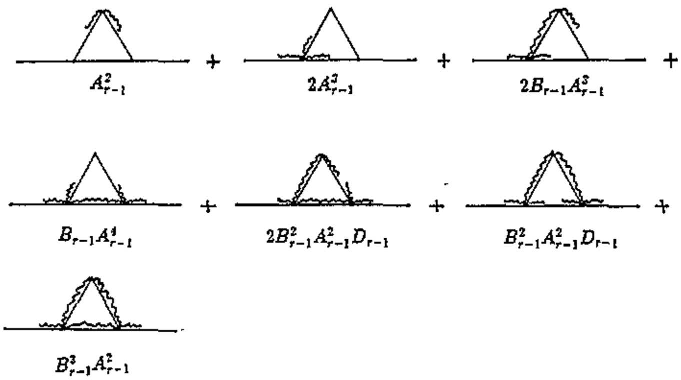

Figure 4. All the possible configurations of S_r are shown. A_{r-1}, B_{r-1} and D_{r-1} are the three restricted generating functions we have defined.

In order to obtain the asymptotic form of $\sum_s n(s)$ we have to study the recursion relations of the restricted generating functions. In figure 5, all possible ways of constructing A_{r+1} are shown. By summing all these contributions we obtain

$$A_{r+1} = (1 + B_r^4)A_r + (B_r + B_r^2)A_r^2 + 3B_r^3 D_r A_r + B_r^2 A_r^3. \tag{9}$$

In the same way we obtain

$$B_{r+1} = B_r^5 + 3B_r^4 D_r + B_r^3 A_r^2 \tag{10}$$

$$D_{r+1} = 2B_r^4 D_r + 8B_r^3 D_r^2 + 3B_r^2 A_r^2 D_r + A_r^2 + 2B_r A_r^3 + 2B_r^2 A_r^3. \tag{11}$$

The initial values (pertinent to the zeroth-stage Koch curve) of these functions are

$$A_0 = D_0 = 1 - p \tag{12}$$

$$B_0 = p \tag{13}$$

where p is the occupied probability of one bond.

Since the recursion equations (9)–(11) are very complicated, it is impossible to obtain the explicit form of A_r, B_r or C_r. Let us restrict ourselves to considering the critical

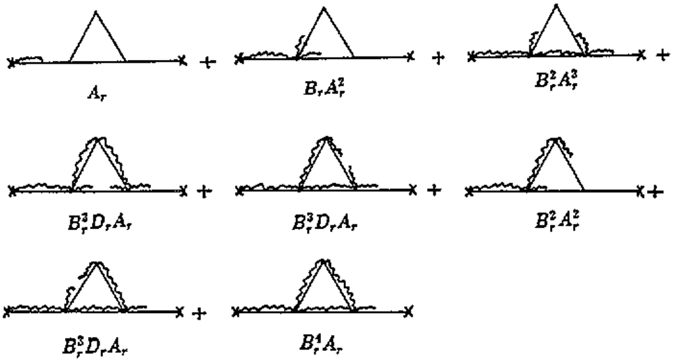

Figure 5. All the possible ways of constructing A_{r+1} are shown.

region only. Following Dhar [14], we introduce an infinitesimal quantity δ characterizing the deviation from critical value $p_c = 1$, and write

$$A_0 = D_0 = \delta \tag{14}$$

$$B_0 = 1 - \delta. \tag{15}$$

Let us choose a small positive number ϵ, which satisfies

$$\delta \ll \epsilon \ll 1 \tag{16}$$

and define the marker r_0 in terms of the relation

$$B_{r_0} = 1 - \epsilon. \tag{17}$$

Then we have two regions: for $r \leqslant r_0$, B_r approximates to 1; and for $r > r_0$, B_r rapidly approaches zero. Hence, for $r \leqslant r_0$, the recursion relations (9) and (11) become

$$A_{r+1} \approx 2A_r \tag{18}$$

$$D_{r+1} \approx 2D_r \tag{19}$$

where terms proportional to second- and higher-orders of δ are neglected.

Considering equations (14), (15), (18) and (19), we have for $r \leqslant r_0$:

$$A_r \approx 2^r \delta \tag{20}$$

$$D_r \approx 2^r \delta. \tag{21}$$

Substituting the expressions (20) and (21) into the recursion relation (10) we obtain

$$B_r \approx 1 - 2^r \delta \qquad \text{for } r \leqslant r_0. \tag{22}$$

Comparing equations (17) and (22), we obtain

$$r_0 = \frac{\ln(\epsilon/\delta)}{\ln 2}. \tag{23}$$

For $r > r_0$, B_r is approximated as zero, and A_r, D_r become

$$A_r \approx A_{r_0} \tag{24}$$

$$D_r \approx A_{r_0}^2. \tag{25}$$

Combining the expressions (7) and (8), and equations (20), (21), (24) and (25), we finally obtain the asymptotic form of $\sum_s n_s$:

$$\sum_s n_s \sim K_1(\epsilon)\delta^2 + K_2(\epsilon)\left(\tfrac{8}{5}\right)^{r_0}\delta^3 + \mathrm{O}(\delta^3). \tag{26}$$

Substituting expression (21) we obtain

$$\sum_s n_s \sim K_1(\epsilon)\delta^2 + K_3(\epsilon)\delta^{\ln 5/\ln 2} + O(\delta^3) \tag{27}$$

where K_1, K_2 and K_3 are all proportional constants depending on ϵ.

We see that the second term is the leading singular term of expression (27) and we then have [1]

$$\alpha = 2 - \frac{\ln 5}{\ln 2}. \tag{28}$$

Therefore, we have obtained the critical exponent α. The critical exponents β and γ can be calculated similarly. From figure 4 we note that for $r \leqslant r_0$, B_r approximates to one; the contributions of the rth-order clusters to $\sum_s sn_s$ and $\sum_s s^2 n_s$ are then proportional to $5^r(5^{-r}S_r)$ and $(5^r)^2(5^{-r}S_r)$ respectively. For $r > r_0$, B_r rapidly approaches zero, and then the dominant contributions to the rth-order clusters are from those structures A^2_{r-1} and $2A^3_{r-1}$ in which an inner vertex is surrounded by clusters with maximum size 5^{r_0}. Thus for $r > r_0$ the contributions of the rth-order clusters to $\sum_s sn_s$ and $\sum_s s^2 n_s$ are proportional to $5^{r_0}(5^{-r}S_r)$ and $(5^{r_0})^2(5^{-r}S_r)$ respectively; we then have

$$\sum_s sn_s \sim \sum_{r\leqslant r_0} S_r + \sum_{r>r_0} 5^{r_0}(5^{-r}S_r) \sim S_{r_0} \tag{29}$$

$$\sum_s s^2 n_s \sim \sum_{r\leqslant r_0} 5^r S_r + \sum_{r>r_0} (5^{r_0})^2(5^{-r}S_r) \sim 5^{r_0} S_{r_0}. \tag{30}$$

Substituting the expressions (8), (20), (21) and (23)–(25) into (29) and (30) we obtain

$$\sum_s sn_s \sim K_4(\epsilon) \tag{31}$$

$$\sum_s s^2 n_s \sim K_5(\epsilon)\delta^{-\ln 5/\ln 2} \tag{32}$$

where K_4 and K_5 are proportional constants depending on ϵ. Then, comparing the expressions (31) and (32) with (2) and (3), one finds that

$$\beta = 0 \tag{33}$$

$$\gamma = \frac{\ln 5}{\ln 2}. \tag{34}$$

We have now calculated the critical exponents α, β and γ with the use of the generating function method. The critical exponent ν can be similarly obtained using the above procedure. In what follows we solve ν by applying the scaling argument [15] rather than using the generating function method.

After a one-step scaling transformation (with scaling factor $b = 3$), the correlation length transforms as follows:

$$\xi(p') = b^{-1}\xi(p) \tag{35}$$

where p' is the renormalized occupied probability of bond. The recursion relation of renormalization group transformation is easily obtained as

$$p' = p^3 + p^4 - p^5. \tag{36}$$

From the above equation we obtain the trivial fixed point $p^* = 1$. Linearizing p' at the vicinity of p_c we can write

$$1 - p' = \lambda(1 - p) \tag{37}$$

where $\lambda = \frac{\partial p'}{\partial p}|_{p*} = 2$. Substituting expressions (35) and (37) into (4), we obtain

$$\nu = \frac{\ln b}{\ln \lambda} = \frac{\ln 3}{\ln 2}. \tag{38}$$

Combining the expressions of α, β, γ and ν, we acquire the scaling law

$$2 - \alpha = \gamma + 2\beta = d_f \nu \tag{39}$$

which is similar to the scaling law satisfied on a regular lattice, except that we substitute the fractal dimension d_f for the Euclidean dimension d.

3. Conclusion

We have studied bond percolation on branching Koch curves. We calculated the critical exponents α, β and γ for the curve of figure 1 by using the generating function method and solved the critical exponent ν in terms of the renormalization-group method, with

$$\alpha = 2 - \frac{\ln 5}{\ln 2} \qquad \beta = 0 \qquad \gamma = \frac{\ln 5}{\ln 2} \qquad \nu = \frac{\ln 3}{\ln 2}.$$

The critical exponents are found to satisfy the scaling law $2 - \alpha = \gamma + 2\beta = d_f \nu$.

A similar calculation can be applied to the other branching Koch curves shown in figure 2 to obtain the critical exponents α, β, γ and ν. We found that the scaling law $2 - \alpha = \gamma + 2\beta = d_f \nu$ is still valid; we conclude that the scaling law is always satisfied for bond percolation on branching Koch curves. We suppose that this conclusion may also be correct for percolation problem on fractals.

Acknowledgments

This work was supported by the National Basic Research project 'Nonlinear science', the National Nature Science Foundation of China and the State Education Committee Grant for doctoral studies.

Appendix A. The derivation of equation (5)

Let us denote the number of sites in an rth-stage branching Koch curve by N_r; one can then easily obtain

$$N_{r+1} = 5N_r - 5 \tag{A1}$$

and

$$N_0 = 2. \tag{A2}$$

Then we have

$$N_r = \tfrac{3}{4}5^r + \tfrac{5}{4}. \tag{A3}$$

Since the whole lattice can always be divided into t_r, rth-stage branching Koch curves, the total number of sites can be written as

$$\begin{aligned} N &= t_r N_r - 5(t_r/5 + t_r/5^2 + \cdots) \\ &= t_r N_r - \tfrac{5}{4} t_r \end{aligned} \tag{A4}$$

where the terms in the bracket come from the consideration that as every five rth-stage Koch curves consist of an $(r+1)$th-stage Koch curve, five sites need to be eliminated. Substituting equation (A.3) into (A.4) we obtain

$$N = \tfrac{3}{4} t_r 5^r . \tag{A5}$$

Substituting equation (A.5) into equation (6) in the text, one finds

$$\sum_s n_s = \tfrac{4}{3} \sum_r 5^{-r} S_r \tag{A6}$$

which is just equation (7) in the text.

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